Antiderivatives and Indefinite integrals

A function $F(x)$ is called the antiderivative of another function $f(x)$ is its derivative is $f(x)$; ie,

$F'(x) = f(x)$. We write

$$\int f(x)\,dx = F(x) + C$$ where C is a constant

integral

The process of finding $F(x)$ is called integration and $F(x) + C$ is called the general antiderivative or the indefinite integral of $f(x)$

For instance, because the derivative of $\cos x$ is $-\sin x$, the antiderivative of "$-\sin x$" written as $\int(-\sin x)\,dx$ is $\cos x + C$, C = any constant

$$\begin{tikzcd} \boxed{\cos x + C} \arrow[r, bend left, "\text{derivative}"] & \boxed{-\sin x} \arrow[l, bend left, "\text{antiderivative}", "\int"'] \end{tikzcd}$$

Ex: Evaluate

$\int 3x^2\,dx$, ie, find $F(x)$ such that $F'(x) = 3x^2$. Clearly, we can recall $F(x) = x^3$ since $(x^3)' = 3x^2$ and we write

$$\boxed{\int 3x^2\,dx = x^3 + C}$$

Rule: Power rule:

$$\int x^n\,dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C, & n \neq -1 \\ \ln|x| + C, & n = -1 \end{cases}$$

Eg: Evaluate

(a) $\int x^3\,dx$; (b) $\int \sqrt{x}\,dx$; (c) $\int x^{2/3}\,dx$

Ans:

Power rule:

(a) $n = 3 \longrightarrow \int x^3\,dx = \frac{1}{4}x^4 + C$ or $\frac{x^4}{4} + C$

Check: $\left(\frac{x^4}{4}\right)' = \frac{4x^{4-1}}{4} = x^3$

(b) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx \quad \left(n = \frac{1}{2},\ n+1 = \frac{3}{2}\right)$

$= \frac{2}{3}x^{3/2} + C$

(c) $\int x^{2/3}\,dx \quad \left(n = 2/3,\ n+1 = \frac{5}{3}\right)$

$\Longrightarrow \frac{3}{5}x^{5/3} + C$

Basic Rules

① $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx\,dx$

② $\int k f(x)\,dx = k \int f(x)\,dx$, k is any constant

Eg: Find the general antiderivative of

ⓐ $y' = 8x - x^2 + 1$

ⓑ $\frac{dy}{dx} = x - \sqrt{x}$

ⓒ $f'(x) = \frac{1 - x^2 + 2x}{x}$

Solution

ⓐ $y = \int y'\,dx = \int (8x - x^2 + 1)\,dx$

$= \int 8x^1\,dx - \int x^2\,dx + \int 1\,dx$

$y = 4x^2 - \frac{1}{3}x^3 + x + C$

ⓑ $y = \int \frac{dy}{dx} = \int (x - \sqrt{x})\,dx$

$= \int x\,dx - \int \sqrt{x}\,dx = \int x\,dx - \int x^{\frac{1}{2}}\,dx$

$= \frac{1}{2}x^2 - \frac{2}{3}x^{3/2} + C$

ⓒ $f(x) = \int f'(x)\,dx = \int \frac{1 - x^2 + 2x}{x}\,dx$

$= \int \left(\frac{1}{x} - \frac{x^2}{x} + \frac{2x}{x}\right)dx$

$= \int x^{-1}\,dx - \int x\,dx + 2\int dx$

$= \ln|x| - \frac{1}{2}x^2 + 2x + C$

Ex: Find the general antiderivative of:

(a) $\theta^4 - \sec^2\theta$

(b) $\cos\theta - \sec\theta\tan\theta$

(c) $e^x - 2^x$

(d) $\frac{1}{t^2} - \frac{1}{\sqrt[3]{t}} + \csc^2 t$

Solutions:

(a) $\int(\theta^4 - \sec^2\theta)\,d\theta = \frac{1}{5}\theta^5 - \tan\theta + C$

(b) $\int(\cos\theta - \sec\theta\tan\theta)\,d\theta = \sin\theta - \sec\theta + C$

(c) $\int(e^x - 2^x)\,dx = e^x - \frac{1}{\ln 2}2^x + C$

(d) $\int\left(\frac{1}{t^2} - \frac{1}{\sqrt[3]{t}} + \csc^2 t\right)dt$

$= \int\left(t^{-2} - t^{-1/3} + \csc^2 t\right)dt = -t^{-1} - \frac{3}{2}t^{2/3} - \cot(t) + C$

Ex: Find the position function of an object which travels with velocity $v(t) = -3t + 4$ with an initial ~~velocity~~ position condition $s(0) = 2\,m$

Solution:

Knowing that the position function

$$s(t) = \int v(t)\,dt = \int (-3t + 4)\,dt$$

$$= -\frac{3}{2}t^2 + 4t + C$$

Initially, $s(0) = 2 \Longrightarrow$

$$s(0) = -\frac{3}{2}(0)^2 + 4(0) + C = 2 \longrightarrow 0 + C = 2$$

$$\longrightarrow C = 2$$

Thus, $s(t) = -\frac{3}{2}t^2 + 4t + 2$

Ex: Find the particular solution

$f'(x) = 8x$ and $f(0) = 5$

Solution:

$$f(x) = \int 8x\,dx = 4x^2 + C$$

Now $f(0)=5 \longrightarrow 4(0) + C = 5 \longrightarrow \boxed{C=5}$

So $\boxed{f(x) = 4x^2 + 5}$

Ex: Find the particular solution

$f''(x) = -1$, $f'(1) = 2$ and $f(2) = 4$

ANS:

$$f'(x) = \int f''(x)\,dx = \int -1\,dx = -x + C$$

Now $f'(1) = 2 \longrightarrow -(1) + C = 2 \longrightarrow C = 3$

So $f'(x) = -x + 3$

Further, $f(x) = \int f'(x)\,dx = \int (-x+3)\,dx$

$$= -\tfrac{1}{2}x^2 + 3x + C$$

Now $f(2) = 4 \longrightarrow -\tfrac{1}{2}(2)^2 + 3(2) + C = 4$

$-2 + 6 + C = 4$

$C = 0$

So $\boxed{f(x) = -\tfrac{1}{2}x^2 + 3x + \underline{\underline{0}}}$